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Analysis of a thermoelastic problem of type III

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Abstract. This paper investigates several aspects of the linear type III thermoelastic theory. First, we consider the most general system of equations for this theory in the case that the conductivity rate is not definite and we prove an existence theorem by means of the semigroups theory. In fact we show that the solutions of the problem generate a quasi-contractive semigroup. Then, assuming that the internal energy is positive definite, the numerical analysis of this problem is performed, by using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. A discrete stability property and a priori error estimates are shown, from which the linear convergence of the algorithm is deduced. Finally, some one- and two-dimensional numerical simulations are presented, for the homogeneous and isotropic case, to demonstrate the accuracy of the approximation and the behaviour of the solution.

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1 Introduction

The classical linear theory of heat conduction predicts that the thermal perturbations at some point in a heat conducting material will be felt instantly, at all other points of the body, however distant. This is usually known as the *paradox of heat conduction*. It is not realistic from a physical point of view since it implies that thermal waves propagate with infinite speed and therefore causality's principle is violated. Many people have been interested to overcome this difficulty and to propose alternative theories which were free of the paradox. We can cite the hyperbolic proposal of Cattaneo and Maxwell for the heat conduction [2]. There are two extensions of the Cattaneo law to the thermoelasticity. One corresponds to the theory of Lord and Shulman [19] and the other is the theory of Green and Lindsay [4]. Good reviews of the several new thermoelastic theories which has been proposed to describe the heat propagation in the second part of the 20th century can be found in [8,9,13,29] and the references cited therein. In this paper we center our attention to the Green and Naghdi proposals. They considered three new thermoelastic theories (see [5–7]), where for two of them the heat conduction does not agree with the usual one. These theories were labelled as type I, II and III, respectively. They are proposed in a rational way and based on an entropy balance law rather than the usual entropy inequality. For the types II and III the thermodynamics makes use of the thermal displacement α that satisfies $\dot{\alpha} = \theta$ where θ is the temperature. Type II thermoelasticity, also known as *thermoelasticity without energy dissipation,* is a limiting case of the type III theory and satisfies that the energy of the system is constant for every time. This last theory is the most general and also contains type I as another particular case. We believe that the mathematical and physical work will reveal the relevance of these new theories. Our work is addressed in this direction.

The theories of Green and Naghdi are currently under study and many research has been developed to understand them (see [10–12,14,15,18,21,22,20,23–28,30], among others). When we consider type II and III theories, it is relevant to note that the system of equations propose new tensors (see equations (1) and (2) below). The majority of the contributions concerning these new theories propose that the *conductivity rate tensor* k_{ij} is positive. Nevertheless from the second law we can only conclude that the thermal conductivity is not negative (see [5, pages 260-261]). But there is no reason *a priori* to impose this condition on the conductivity rate. It is only because the mathematical study of the equations which suggests that, if we want to have an stable problem, we need to impose the positivity of this new

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tensor. Therefore, we believe that it is suitable to study this problem without the assumption of the positivity of this tensor (see [16,17]). We here restrict our attention to the type III theory.

In this paper we are going to consider an aspect which has not been studied previously. This is the existence of the solutions in the case that the internal energy may not be positive. We will prove the existence of the solutions under suitable conditions by means of the semigroup of linear operators argument. As a consequence we will obtain an uniqueness and continuous dependence result. In particular, the new continuous dependence result improves the one proposed in [16]. Later we will develop a numerical study to this problem which, in this new perspective, we will assume the positivity of the conductivity rate and the internal energy to simplify the calculations. We will use the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. A discrete stability result will be proved and a priori error estimates will be provided, from which we will derive the linear convergence of the algorithm under suitable additional regularity conditions. Finally, we will perform some numerical simulations in one and two dimensions assuming that all the tensors are homogeneous and isotropic.

2 Preliminaries

The most general system of field equations for the type III thermoelasticity takes the form:

$$\rho\ddot{u}_i = \big(a_{ijkh}u_{k,h} - a_{ij}\theta + G_{ijr}\alpha_{,r}\big)_{,j} + \rho f_i, \tag{1}$$

$$c\ddot{\alpha} = -a_{ij}\dot{u}_{i,j} + b_i\theta_{,i} + \big(G_{ijr}u_{i,j} + b_r\theta + k_{rj}\alpha_{,j} + b_{rj}\theta_{,j}\big)_{,r} + cS, \tag{2}$$

where $\boldsymbol{u} = (u_i)_{i=1}^d$ is the displacement field, θ is the temperature, ρ is the mass density, c is the heat capacity, a_{ijkh} is the elasticity tensor, a_{ij} is the thermal coupling tensor (which is related to the thermal displacement tensor), b_{ij} is the thermal conductivity tensor, α is the thermal displacement which is defined by

$$\alpha(\boldsymbol{x}, t) = \int_0^t \theta(\boldsymbol{x}, s)\, ds + \alpha_0(\boldsymbol{x}), \tag{3}$$

k_{ij} is a tensor which is typical of types II and III Green-Naghdi's theories (usually called conductivity rate), and the tensors G_{ijr} and b_i are also tensors which appear in the types II and III theories but only in case that the materials are no isotropic neither centrosymmetric. It is worth recalling that the symmetries

$$a_{ijkh} = a_{khij} \tag{4}$$

and

$$k_{ik} = k_{ki},\ b_{ik} = b_{ki} \tag{5}$$

are satisfied.

In the next section we will prove an existence theorem without imposing any sign on the conductivity rate. To this end we will use the semigroup of linear operators theory.

In this paper we are going to study the solutions to system (1)-(2) in a bounded domain B contained in $\mathbb{R}^d$, $d = 2, 3$, such that its boundary Γ is smooth enough to apply the Divergence theorem. Moreover, we will study it in a time interval $[0, T]$, where $T > 0$ is the final time.

To fully describe the problem we must consider some boundary and initial conditions. We assume the boundary conditions, for $i = 1, \dots, d$,

$$u_i(\boldsymbol{x}, t) = 0,\ \alpha(\boldsymbol{x}, t) = 0,\ \boldsymbol{x} \in \Gamma, \tag{6}$$

and the initial conditions, for $i = 1, \dots, d$ and $\boldsymbol{x} \in B$,

$$u_i(\boldsymbol{x}, 0) = u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = v_i^0(\boldsymbol{x}), \quad \alpha(\boldsymbol{x}, 0) = \alpha^0(\boldsymbol{x}), \quad \dot{\alpha}(\boldsymbol{x}, 0) = \theta^0(\boldsymbol{x}). \tag{7}$$

3 Existence of solutions

The main goal of this section is to prove an existence, uniqueness and continuous dependence result for the solutions to the problem defined by (1), (2), (6) and (7). Therefore, we will impose the following assumptions:

(A1) The mass density $\rho(\boldsymbol{x})$ and the heat capacity $c(\boldsymbol{x})$ are strictly positive functions.
(A2) The thermal conductivity is positive definite. That is, there exists a positive constant C such that

$$b_{ij}\xi_i\xi_j \geq C\xi_i\xi_i, \tag{8}$$

for every vector (ξ_i).

(A3) There exists a positive constant C such that

$$\int_B a_{ijkl}\, u_{i,j}\, u_{k,l}\, dv \geq C \int_B u_{i,j}\, u_{i,j}\, dv \tag{9}$$

for every (u_i) and θ which vanish at the boundary.

(A4) All the tensors a_{ijkl}, b_{ijr}, a_{ij}, b_{ij}, k_{ij}, G_{ijr}, b_i are bounded.

The physical meaning of condition (A1) is obvious. Condition (A3) is related to the positivity of the elastic energy and may be interpreted with the help of the theory of mechanical stability. Condition (A2) guarantees the dissipation of the energy for the system. It corresponds to toughen the consequences of the second law.

It is worth noting that we do not assume in general any positivity for the internal energy which is defined by

$$a_{ijhk}u_{k,h}u_{i,j} + k_{ik}\alpha_{,k}\alpha_{,i} + 2G_{ijr}u_{i,j}\alpha_{,r}.$$

We now transform the boundary- initial- value problem determined by our system and the initial and boundary conditions into an abstract problem on a suitable Hilbert space. We denote

$$\mathcal{Z} = \{(\boldsymbol{u}, \boldsymbol{v}, \alpha, \theta), \boldsymbol{u} \in \mathbf{W}_0^{1,2}(B), \alpha \in W_0^{1,2}(B), \boldsymbol{v} \in \mathbf{L}^2(B), \theta \in L^2(B)\},$$

where $W_0^{1,2}$ and L^2 are the usual Sobolev spaces and $\mathbf{W}_0^{1,2} = [W_0^{1,2}]^d$, $\mathbf{L}^2 = [L^2]^d$.

We denote by

$$\begin{aligned}
&A_i\mathbf{u} = \rho^{-1}(a_{ijrs}u_{r,s})_{,j}, \quad B_i\theta = -\rho^{-1}(a_{ij}\theta)_{,j},\\
&E\mathbf{v} = -c^{-1}a_{ij}v_{i,j}, \quad F\alpha = c^{-1}(k_{ij}\alpha_{,j})_{,i},\\
&G\theta = c^{-1}[(b_{ij}\theta_{,j})_{,i} + b_i\theta_{,i} + (b_r\theta)_{,r}],\\
&C_i(\alpha) = \rho^{-1}(G_{ijr}\alpha_{,r})_{,j}, \quad D(\mathbf{u}) = c^{-1}(G_{ijr}u_{i,j})_{,r},\\
&\mathbf{A} = (A_i), \quad \mathbf{B} = (B_i), \quad \mathbf{C} = (C_i),
\end{aligned}$$

and

$$\mathcal{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{0} & \mathbf{C} & \mathbf{B} \\ 0 & 0 & 0 & I \\ D & E & F & G \end{pmatrix}, \tag{10}$$

where $\mathbf{I}$ and I are the identity operators in the respective spaces.

Our boundary initial-value problem can be transformed into the following abstract equation in the Hilbert space $\mathcal{Z}$:

$$\frac{d\varpi}{dt} = \mathcal{A}\varpi(t) + \mathcal{F}(t), \quad \varpi(0) = \varpi_0, \tag{11}$$

where

$$\mathcal{F}(t) = (\mathbf{0}, \mathbf{f}, 0, S), \quad \varpi_0 = (\mathbf{u}^0, \mathbf{v}^0, \alpha^0, \theta^0). \tag{12}$$

It is worth noting that we can define the inner product in the Hilbert space in the following way. If we assume that $\varpi = (\boldsymbol{u}, \boldsymbol{v}, \alpha, \theta)$ and $\varpi^* = (\boldsymbol{u}^*, \boldsymbol{v}^*, \alpha^*, \theta^*)$ we consider

$$\langle \varpi, \varpi^* \rangle = \frac{1}{2} \int_B (\rho v_i v_i^* + c\theta\theta^* + C_{ijkl}u_{i,j}u_{k,l}^* + G_{ijr}(u_{i,j}\alpha_{,r}^* + u_{i,j}^*\alpha_{,r}) + \epsilon_0\alpha_{,i}\alpha_{,i}^*)dv,$$

where ϵ_0 is a sufficiently large positive number to guarantee that our inner product is equivalent to the usual one in $\mathcal{Z}$. We note that

$$||\varpi||^2 = \frac{1}{2} \int_B (\rho v_i v_i + c\theta^2 + C_{ijkl}u_{i,j}u_{k,l} + 2G_{ijr}u_{i,j}\alpha_{,r} + \epsilon_0\alpha_{,i}\alpha_{,i})dv.$$

Hence we could select ϵ_0 in terms of the tensors G_{ijk} and C_{ijkl}. For instance in case that $G_{ijk}u_{i,j}\alpha_{,k} \leq M(u_{i,j}u_{i,j} + \alpha_{,k}\alpha_{,k})$ we can select ϵ_0 such that $C\epsilon_0 > M^2$ where C is given in the assumption (A3).

First, we point out that the domain $\mathcal{D}$ of the operator $\mathcal{A}$ is determined by the elements in the Hilbert space such that $\mathbf{v} \in \mathbf{W}_0^{1,2}(B),\ \theta \in W_0^{1,2}(B), \mathbf{A}\mathbf{v} + \mathbf{C}\alpha \in \mathbf{L}^2(B), D\mathbf{u} + F\alpha + G\theta \in L^2$. In particular we note that

$$\mathbf{W}_0^{1,2}(B) \cap \mathbf{W}^{2,2}(B) \times \mathbf{W}_0^{1,2}(B) \times W_0^{1,2}(B) \cap W^{2,2}(B) \times W_0^{1,2}(B) \cap W^{2,2}(B)$$

is contained in the domain. Therefore, it is clear that the domain is dense in the Hilbert space $\mathcal{Z}$.

Lemma 1 *There exists a positive constant δ such that, for all $\varpi \in \mathcal{D}$, we have*

$$\langle \mathcal{A}\varpi, \varpi \rangle \leq \delta ||\varpi||^2. \tag{13}$$

Proof By using the evolution equations and the divergence theorem and taking into account the boundary conditions, we obtain

$$\langle \mathcal{A}\varpi, \varpi \rangle = \frac{1}{2} \int_B (\epsilon_0 \alpha_{,i} \theta_{,i} - k_{ij} \alpha_{,i} \theta_{,j} - b_{ij} \theta_{,i} \theta_{,j}) dv.$$

The use of the Holder and the arithmetic-geometric mean inequality allow us to see that

$$\langle \mathcal{A}\varpi, \varpi \rangle \leq M_1 \int_B \alpha_{,i} \alpha_{,i} dv + \frac{1}{2}(\epsilon_1 + \epsilon_2 - 1) \int_B b_{ij} \theta_{,i} \theta_{,j} dv,$$

where ϵ_1, ϵ_2 are two positive numbers, but as small as we want, and M_1 is a calculable positive constant depending on $\epsilon_0, \epsilon_1, \epsilon_2, k_{ij}$ and b_{ij}. If we impose that $\epsilon_1 + \epsilon_2 < 1$ and we recall the definition of the inner product, the lemma is proved.

Lemma 2 *There exists a positive constant λ_0 such that the operator $\mathcal{A}$ satisfies the range condition:*

$$Range\,(\lambda_0 \mathcal{ID} - \mathcal{A}) = \mathcal{Z}. \tag{14}$$

Proof Let $\varpi^* = (\boldsymbol{u}^*, \boldsymbol{v}^*, \alpha^*, \theta^*) \in \mathcal{Z}$. We must prove that the equation

$$\lambda_0 \varpi - \mathcal{A}\varpi = \varpi^*, \tag{15}$$

has a solution $\varpi = (\boldsymbol{u}, \boldsymbol{v}, \alpha, \theta) \in \mathcal{D}(\mathcal{A})$ for λ_0 so large. From the definition of $\mathcal{A}$, we obtain the system:

$$\begin{aligned}
&\lambda_0 \boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u}^*,\\
&\lambda_0 \boldsymbol{v} - \mathbf{A}\boldsymbol{u} - \mathbf{C}\alpha - \mathbf{B}\theta = \boldsymbol{v}^*,\\
&\lambda_0 \alpha - \theta = \alpha^*,\\
&\lambda_0 \theta - D\boldsymbol{u} - E\boldsymbol{v} - F\alpha - G\theta = \theta^*.
\end{aligned}$$

By substitution of the first and third equations into the others, we get

$$\begin{aligned}
&\lambda_0^2 \boldsymbol{u} - \mathbf{A}\boldsymbol{u} - \mathbf{C}\alpha - \lambda_0 \mathbf{B}\alpha = \lambda_0 \boldsymbol{u}^* + \boldsymbol{v}^* - \mathbf{B}\alpha^*,\\
&\lambda_0^2 \alpha - D\boldsymbol{u} - F\alpha - \lambda_0 G\alpha - \lambda_0 E\boldsymbol{u} = \lambda_0 \alpha^* - G\alpha^* - E\boldsymbol{u}^*.
\end{aligned}$$

To study our new system we introduce a bilinear form $\mathcal{B}_{\lambda_0}$ on $\mathbf{W}_0^{1,2} \times W_0^{1,2}$ defined as

$$\mathcal{B}_{\lambda_0}\Big[(\boldsymbol{u}, \alpha), (\tilde{\boldsymbol{u}}, \tilde{\alpha})\Big] =< \begin{pmatrix} \lambda_0^2 - \mathbf{A} & -\lambda_0 \mathbf{B} - \mathbf{C} \\ -\lambda_0 E - D & \lambda_0^2 - F - \lambda_0 G \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \alpha \end{pmatrix}, (\tilde{\boldsymbol{u}}, \tilde{\alpha}) >_{\mathbf{L} \times L^2}. \tag{16}$$

After use of the divergence theorem we see that this bilinear form is bounded on $\mathbf{W}_0^{1,2} \times W_0^{1,2}$ for every positive λ_0. Using both the divergence theorem and the arithmetic-geometric mean inequality we can prove that $\mathcal{B}_{\lambda_0}$ (for λ_0 sufficiently large) is coercive in $\mathbf{W}_0^{1,2} \times W_0^{1,2}$. The right hand-side of the system lies in $\mathbf{W}^{-1} \times W^{-1}$ which is the dual of $\mathbf{W}_0^{1,2} \times W_0^{1,2}$. Hence Lax- Milgram theorem implies the existence of a solution to the system for $\boldsymbol{u}$ and α. If we return to our system, we also obtain the solution for $\boldsymbol{v}$ and θ and the lemma is proved.

The use of these two lemmas and the Lummer-Phillips corollary to the Hille-Yosida theorem lead to the next theorem.

Theorem 1 *The operator $\mathcal{A}$ generates a quasi-contractive semigroup in $\mathcal{Z}$.*

Thus, we conclude the following existence and uniqueness result.

Theorem 2 *Let us assume that the conditions (A1) -(A4) are satisfied and the supply terms have the additional regularity*

$$\mathbf{f} \in C^1([0,T]; \mathbf{L}^2(B)) \cap C^0([0,T]; \mathbf{W}_0^{1,2}(B)), \quad S \in C^1([0,T]; L^2(B)) \cap C^0([0,T]; W_0^{1,2}(B)).$$

Then, for any $(\boldsymbol{u}^0, \boldsymbol{v}^0, \alpha^0, \theta^0)$ in $\mathcal{D}$, there exists a unique solution to the evolution equations; namely, there exists a unique $(\boldsymbol{u}(t), \boldsymbol{v}(t), \alpha(t), \theta(t)) \in C^1([0,T]; \mathcal{Z}) \cap C^0([0,T]; \mathcal{D})$.

This theorem states that our problem is well posed in the sense of Hadamard. In particular, we have the following estimate to the solutions:

$$||(\boldsymbol{u}(t), \boldsymbol{v}(t), \alpha(t), \theta(t))|| \leq \exp(\delta t)\Big(||(\boldsymbol{u}^0, \boldsymbol{v}^0, \alpha^0, \theta^0)|| + \int_0^t \Big(\int_B (f_i f_i + S^2) dv\Big)^{1/2}\Big). \tag{17}$$

It is worth noting that in the case that the conductivity rate tensor is positive definite and we assume the existence of a positive constant C such that

$$\int_B (a_{ijhk} u_{k,h} u_{i,j} + k_{ik} \alpha_{,k} \alpha_{,i} + 2G_{ijr} u_{i,j} \alpha_{,r}) dv \geq C \int_B (u_{i,j} u_{i,j} + \alpha_{,k} \alpha_{,k}) dv, \tag{18}$$

for every $u_{i,j}$ and $\alpha_{,k}$, the arguments to prove the existence become easier. We first note that in this case we can define in the Hilbert space the inner product:

$$\langle \varpi, \varpi^* \rangle = \frac{1}{2} \int_B (\rho v_i v_i^* + c\theta\theta^* + C_{ijkl} u_{i,j} u_{k,l}^* + G_{ijr}(u_{i,j} \alpha_{,r}^* + u_{i,j}^* \alpha_{,r}) + k_{ij} \alpha_{,i} \alpha_{,j}^*) dv.$$

In fact, we can prove that, for every $\varpi \in \mathcal{D}$,

$$\langle \mathcal{A}\varpi, \varpi \rangle \leq 0, \tag{19}$$

and by means of an argument very similar to the proof of Lemma 2, we also have that zero belongs to the resolvent of the operator $\mathcal{A}$. As a consequence, the semigroup is contractive because the operator $\mathcal{A}$ is dissipative. That is, we can obtain an inequality of the type

$$||(\boldsymbol{u}(t), \boldsymbol{v}(t), \alpha(t), \theta(t))|| \leq \Big(||(\boldsymbol{u}^0, \boldsymbol{v}^0, \alpha^0, \theta^0)|| + \int_0^t \Big(\int_B (f_i f_i + S^2) dv\Big)^{1/2}\Big). \tag{20}$$

It is worth noting that several studies on the asymptotic behaviour have been developed in the case of centrosymmetric materials; however, we do not know any study on the asymptotic behavior in case that the tensors G_{ijr} and b_i are different from zero.

In order to provide the numerical approximation, we will derive the variational form of the above thermoelastic problem (1), (2), (6) and (7).

Let $Y = L^2(B)$, $H = [L^2(B)]^d$, $Q = [L^2(B)]^{d\times d}$ and define the variational spaces $E = W_0^{1,2}(B)$ and $V = \mathbf{W}_0^{1,2}(B)$.

Let us define the following operators for the sake of simplicity in the writing:

$$\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v}) &= \int_B a_{ijkh} u_{k,h} v_{i,j}\, d\boldsymbol{x}, \quad \forall \boldsymbol{u} = (u_i)_{i=1}^d,\ \boldsymbol{v} = (v_i)_{i=1}^d \in V,\\
G(\alpha, \boldsymbol{v}) &= \int_B G_{ijr} \alpha_{,r} v_{i,j}\, d\boldsymbol{x}, \quad \forall \alpha \in E,\ \boldsymbol{v} = (v_i)_{i=1}^d \in V,\\
b(\theta, \eta) &= \int_B b_{rj} \theta_{,j} \eta_{,r}\, d\boldsymbol{x}, \quad \forall \theta,\ \eta \in E,\\
A(\theta, \boldsymbol{v}) &= \int_B \theta\, a_{ij} v_{i,j}\, d\boldsymbol{x}, \quad \forall \theta \in E,\ \boldsymbol{v} = (v_i)_{i=1}^d \in V,\\
K(\alpha, \eta) &= \int_B k_{rj} \alpha_{,j}\, \eta_{,r} d\boldsymbol{x}, \quad \forall \alpha,\ \eta \in E,\\
B(\theta, \eta) &= \int_B 2b_i \theta_{,i} \eta\, d\boldsymbol{x}, \quad \forall \theta,\ \eta \in E.
\end{aligned}$$

Using now Green's formula and boundary conditions (6) we obtain the following variational formulation.

Problem VP. *Find the velocity field* $\boldsymbol{v} : [0, T] \to V$ *and the temperature* $\theta : [0, T] \to E$ *such that* $\boldsymbol{v}(0) = \boldsymbol{v}^0$, $\theta(0) = \theta^0$, *and, for a.e.* $t \in (0, T)$ *and for all* $\boldsymbol{w} \in V$, $\eta \in E$,

$$\rho(\dot{\boldsymbol{v}}(t), \boldsymbol{w})_H + a(\boldsymbol{u}(t), \boldsymbol{w}) = \rho(\boldsymbol{f}(t), \boldsymbol{w})_H + A(\theta(t), \boldsymbol{w}) - G(\alpha(t), \boldsymbol{w}), \tag{21}$$

$$c(\dot{\theta}(t), \eta)_Y + b(\theta(t), \eta) + K(\alpha(t), \eta) = (c\, S(t), \eta)_Y - A(\eta, \boldsymbol{v}(t)) - G(\eta, \boldsymbol{u}(t)) + B(\theta(t), \eta), \tag{22}$$

where the displacement and thermal displacement fields are then recovered from the relations

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\, ds + \boldsymbol{u}^0, \quad \alpha(t) = \int_0^t \theta(s)\, ds + \alpha^0. \tag{23}$$

4 Fully discrete approximations: an a priori error analysis

In this section, we now consider a fully discrete approximation of Problem VP. This is done in two steps. First, we assume that the domain $\overline{B}$ is polyhedral and we denote by $\mathcal{T}^h$ a regular triangulation in the sense of [3]. Thus, we construct the finite dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by

$$V^h = \{\boldsymbol{z}^h \in [C(\overline{B})]^d \; ; \; \boldsymbol{z}^h_{|Tr} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h, \quad \boldsymbol{z}^h = \boldsymbol{0} \quad \text{on} \quad \Gamma\}, \tag{24}$$

$$E^h = \{\eta^h \in C(\overline{B}) \; ; \; \eta^h_{|Tr} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad \eta^h = 0 \quad \text{on} \quad \Gamma\}, \tag{25}$$

where $P_1(Tr)$ represents the space of polynomials of degree less or equal to one in the element Tr, i.e. the finite element spaces V^h and E^h are composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\boldsymbol{u}^{0h}$, $\boldsymbol{v}^{0h}$, θ^{0h} and α^{0h} are given by

$$\boldsymbol{u}^{0h} = \mathcal{P}^h_1 \boldsymbol{u}^0, \quad \boldsymbol{v}^{0h} = \mathcal{P}^h_1 \boldsymbol{v}^0, \quad \theta^{0h} = \mathcal{P}^h_2 \theta^0, \quad \alpha^{0h} = \mathcal{P}^h_2 \alpha^0, \tag{26}$$

where $\mathcal{P}^h_1$ and $\mathcal{P}^h_2$ are the classical finite element interpolation operators over V^h and E^h, respectively (see, e.g., [3]).

Secondly, we consider a partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. In this case, we use a uniform partition with step size $\tau = T/N$ and nodes $t_n = n\,\tau$ for $n = 0, 1, \ldots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/\tau$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Problem VP$^{h\tau}$. *Find the discrete velocity field $\boldsymbol{v}^{h\tau} = \{\boldsymbol{v}^{h\tau}_n\}_{n=0}^N \subset V^h$ and the discrete temperature $\theta^{h\tau} = \{\theta^{\tau h}_n\}_{n=0}^N \subset E^h$ such that $\boldsymbol{v}^{h\tau}_0 = \boldsymbol{v}^{0h}$, $\theta^{h\tau}_0 = \theta^{0h}$, and, for $n = 1, \ldots, N$ and for all $\boldsymbol{w}^h \in V^h$, $\eta^h \in E^h$,*

$$\rho(\delta \boldsymbol{v}^{h\tau}_n, \boldsymbol{w}^h)_H + a(\boldsymbol{u}^{h\tau}_n, \boldsymbol{w}^h) = \rho(\boldsymbol{f}_n, \boldsymbol{w}^h)_H + A(\theta^{h\tau}_n, \boldsymbol{w}^h) - G(\alpha^{h\tau}_n, \boldsymbol{w}^h), \tag{27}$$

$$c(\delta\theta^{h\tau}_n, \eta^h)_Y + b(\theta^{h\tau}_n, \eta^h) + K(\alpha^{h\tau}_n, \eta^h) = (c\, S_n, \eta^h)_Y - A(\eta^h, \boldsymbol{v}^{h\tau}_n) - G(\eta^h, \boldsymbol{u}^{h\tau}_n) + B(\theta^{h\tau}_n, \eta^h), \tag{28}$$

where the discrete displacement and thermal displacement fields are then recovered from the relations

$$\boldsymbol{u}^{h\tau}_n = \tau \sum_{j=1}^n \boldsymbol{v}^{h\tau}_j + \boldsymbol{u}^{0h}, \quad \alpha^{h\tau}_n = \tau \sum_{j=1}^n \theta^{h\tau}_j + \alpha^{0h}. \tag{29}$$

We note that the existence of a unique discrete solution to Problem $VP^{h\tau}$ is obtained in a straightforward way using the classical Lax-Milgram lemma and assumptions (A1)-(A4) and (18).

Remark 1 We note that condition (18) is the usual condition required in thermoelastic problems. For instance, if we assume that the body is homogeneous and isotropic, then constitutive equations (1) become

$$\begin{aligned} &\rho\ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu) u_{j,ji} - a\dot{\alpha}_{,i} + \rho f_i, \\ &c\ddot{\alpha} = -a\dot{u}_{i,i} + k\alpha_{,ii} + b\dot{\alpha}_{,ii} + cS. \end{aligned} \tag{30}$$

Therefore, proceeding as in other studies the required conditions are found using, for instance, the following assumptions on the constitutive coefficients:

$$\rho > 0, \quad \mu > 0, \quad 3\lambda + 2\mu > 0, \quad c > 0, \quad k > 0, \quad b > 0. \tag{31}$$

First, we have the following stability property.

Lemma 3 *Let the assumptions of Theorem 2 hold. Under additional assumption (18), it follows that the sequences $\{\boldsymbol{u}^{h\tau}, \boldsymbol{v}^{h\tau}, \theta^{h\tau}, \alpha^{h\tau}\}$ generated by Problem $VP^{h\tau}$ satisfy the stability estimate:*

$$\|\boldsymbol{v}^{h\tau}_n\|^2_H + \|\nabla \boldsymbol{u}^{h\tau}_n\|^2_Q + \|\nabla \alpha^{h\tau}_n\|^2_H + \|\theta^{h\tau}_n\|^2_Y \le C,$$

where C is a positive constant which is independent of the discretization parameters h and τ.

Proof Taking $\boldsymbol{v}^{h\tau}_n$ as a test function in variational equation (27) we find that

$$\rho(\delta \boldsymbol{v}^{h\tau}_n, \boldsymbol{v}^{h\tau}_n)_H + a(\boldsymbol{u}^{h\tau}_n, \boldsymbol{v}^{h\tau}_n) = \rho(\boldsymbol{f}_n, \boldsymbol{v}^{h\tau}_n)_H + A(\theta^{h\tau}_n, \boldsymbol{v}^{h\tau}_n) - G(\alpha^{h\tau}_n, \boldsymbol{v}^{h\tau}_n).$$

Keeping in mind that

$$\begin{aligned}
&(\delta \boldsymbol{v}_n^{h\tau}, \boldsymbol{v}_n^{h\tau})_H \geq \frac{1}{2\tau}\left\{\|\boldsymbol{v}_n^{h\tau}\|_H^2 - \|\boldsymbol{v}_{n-1}^{h\tau}\|_H^2\right\},\\
&a(\boldsymbol{u}_n^{h\tau}, \boldsymbol{v}_n^{h\tau}) = \frac{1}{2\tau}\Big\{a(\boldsymbol{u}_n^{h\tau}, \boldsymbol{u}_n^{h\tau}) - a(\boldsymbol{u}_{n-1}^{h\tau}, \boldsymbol{u}_{n-1}^{h\tau}) + a(\boldsymbol{u}_n^{h\tau} - \boldsymbol{u}_{n-1}^{h\tau}, \boldsymbol{u}_n^{h\tau} - \boldsymbol{u}_{n-1}^{h\tau})\Big\},\\
&|(\boldsymbol{f}_n, \boldsymbol{v}_n^{h\tau})_H| \leq C\|\boldsymbol{v}_n^{h\tau}\|_H,
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{\rho}{2\tau}\left\{\|\boldsymbol{v}_n^{h\tau}\|_H^2 - \|\boldsymbol{v}_{n-1}^{h\tau}\|_H^2\right\} + \frac{1}{2\tau}\Big\{a(\boldsymbol{u}_n^{h\tau}, \boldsymbol{u}_n^{h\tau}) - a(\boldsymbol{u}_{n-1}^{h\tau}, \boldsymbol{u}_{n-1}^{h\tau}) + a(\boldsymbol{u}_n^{h\tau} - \boldsymbol{u}_{n-1}^{h\tau}, \boldsymbol{u}_n^{h\tau} - \boldsymbol{u}_{n-1}^{h\tau})\Big\}\\
&\qquad \leq C\|\boldsymbol{v}_n^{h\tau}\|_H + A(\theta_n^{h\tau}, \boldsymbol{v}_n^{h\tau}) - G(\alpha_n^{h\tau}, \boldsymbol{v}_n^{h\tau}).
\end{aligned} \tag{32}$$

Now, taking $\theta_n^{h\tau}$ as a test function in variational equation (28) we find that

$$c(\delta\theta_n^{h\tau}, \theta_n^{h\tau})_Y + b(\theta_n^{h\tau}, \theta_n^{h\tau}) + K(\alpha_n^{h\tau}, \theta_n^{h\tau}) = (c\, S_n, \theta_n^{h\tau})_Y - A(\theta_n^{h\tau}, \boldsymbol{v}_n^{h\tau}) - G(\theta_n^{h\tau}, \boldsymbol{u}_n^{h\tau}) + B(\theta_n^{h\tau}, \theta_n^{h\tau}).$$

Taking into account that

$$\begin{aligned}
&(\delta\theta_n^{h\tau}, \theta_n^{h\tau})_Y \geq \frac{1}{2\tau}\left\{\|\theta_n^{h\tau}\|_Y^2 - \|\theta_{n-1}^{h\tau}\|_Y^2\right\},\\
&b(\theta_n^{h\tau}, \theta_n^{h\tau}) \geq C\|\nabla\theta_n^{h\tau}\|_H^2,\\
&K(\alpha_n^{h\tau}, \theta_n^{h\tau}) = \frac{1}{2\tau}\Big\{K(\alpha_n^{h\tau}, \alpha_n^{h\tau}) - K(\alpha_{n-1}^{h\tau}, \alpha_{n-1}^{h\tau}) + K(\alpha_n^{h\tau} - \alpha_{n-1}^{h\tau}, \alpha_n^{h\tau} - \alpha_{n-1}^{h\tau})\Big\},\\
&B(\theta_n^{h\tau}, \theta_n^{h\tau}) \leq C\|\theta_n^{h\tau}\|_Y^2 + \epsilon\|\nabla\theta_n^{h\tau}\|_H^2,\\
&|(S_n, \theta_n^{h\tau})_Y| \leq C\|\theta_n^{h\tau}\|_Y,
\end{aligned}$$

where $\epsilon > 0$ is assumed small enough, we obtain

$$\begin{aligned}
&\frac{c}{2\tau}\left\{\|\theta_n^{h\tau}\|_Y^2 - \|\theta_{n-1}^{h\tau}\|_Y^2\right\} + \frac{1}{2\tau}\Big\{K(\alpha_n^{h\tau}, \alpha_n^{h\tau}) - K(\alpha_{n-1}^{h\tau}, \alpha_{n-1}^{h\tau}) + K(\alpha_n^{h\tau} - \alpha_{n-1}^{h\tau}, \alpha_n^{h\tau} - \alpha_{n-1}^{h\tau})\Big\}\\
&\qquad \leq C\|\theta_n^{h\tau}\|_Y + C\|\theta_n^{h\tau}\|_Y^2 - A(\theta_n^{h\tau}, \boldsymbol{v}_n^{h\tau}) - G(\theta_n^{h\tau}, \boldsymbol{u}_n^{h\tau}).
\end{aligned} \tag{33}$$

Combining estimates (32) and (33) it follows that

$$\begin{aligned}
&\frac{\rho}{2\tau}\left\{\|\boldsymbol{v}_n^{h\tau}\|_H^2 - \|\boldsymbol{v}_{n-1}^{h\tau}\|_H^2\right\} + \frac{1}{2\tau}\Big\{a(\boldsymbol{u}_n^{h\tau}, \boldsymbol{u}_n^{h\tau}) - a(\boldsymbol{u}_{n-1}^{h\tau}, \boldsymbol{u}_{n-1}^{h\tau}) + a(\boldsymbol{u}_n^{h\tau} - \boldsymbol{u}_{n-1}^{h\tau}, \boldsymbol{u}_n^{h\tau} - \boldsymbol{u}_{n-1}^{h\tau})\Big\}\\
&\qquad + \frac{1}{2\tau}\left\{\|\theta_n^{h\tau}\|_Y^2 - \|\theta_{n-1}^{h\tau}\|_Y^2\right\} + \frac{1}{2\tau}\Big\{K(\alpha_n^{h\tau}, \alpha_n^{h\tau}) - K(\alpha_{n-1}^{h\tau}, \alpha_{n-1}^{h\tau}) + K(\alpha_n^{h\tau} - \alpha_{n-1}^{h\tau}, \alpha_n^{h\tau} - \alpha_{n-1}^{h\tau})\Big\}\\
&\qquad + G(\alpha_n^{h\tau}, \boldsymbol{v}_n^{h\tau}) + G(\theta_n^{h\tau}, \boldsymbol{u}_n^{h\tau})\\
&\quad \leq C(\|\boldsymbol{v}_n^{h\tau}\|_H + \|\theta_n^{h\tau}\|_Y + \|\theta_n^{h\tau}\|_Y^2).
\end{aligned}$$

Observing that

$$G(\alpha_n^{h\tau}, \boldsymbol{v}_n^{h\tau}) + G(\boldsymbol{u}_n^{h\tau}, \theta_n^{h\tau}) = \frac{1}{k}\Big\{G(\alpha_n^{h\tau}, \boldsymbol{u}_n^{h\tau}) - G(\alpha_{n-1}^{h\tau}, \boldsymbol{u}_{n-1}^{h\tau}) + G(\alpha_n^{h\tau} - \alpha_{n-1}^{h\tau}, \boldsymbol{u}_n^{h\tau} - \boldsymbol{u}_{n-1}^{h\tau})\Big\},$$

and that using assumption (18) we have

$$a(\boldsymbol{u}_n^{h\tau} - \boldsymbol{u}_{n-1}^{h\tau}, \boldsymbol{u}_n^{h\tau} - \boldsymbol{u}_{n-1}^{h\tau}) + K(\alpha_n^{h\tau} - \alpha_{n-1}^{h\tau}, \alpha_n^{h\tau} - \alpha_{n-1}^{h\tau}) + 2G(\alpha_n^{h\tau} - \alpha_{n-1}^{h\tau}, \boldsymbol{u}_n^{h\tau} - \boldsymbol{u}_{n-1}^{h\tau}) \geq 0,$$

it follows that

$$\begin{aligned}
&\frac{\rho}{2\tau}\left\{\|\boldsymbol{v}_n^{h\tau}\|_H^2 - \|\boldsymbol{v}_{n-1}^{h\tau}\|_H^2\right\} + \frac{1}{2\tau}\Big\{a(\boldsymbol{u}_n^{h\tau}, \boldsymbol{u}_n^{h\tau}) - a(\boldsymbol{u}_{n-1}^{h\tau}, \boldsymbol{u}_{n-1}^{h\tau})\Big\} + \frac{1}{k}\Big\{G(\alpha_n^{h\tau}, \boldsymbol{u}_n^{h\tau}) - G(\alpha_{n-1}^{h\tau}, \boldsymbol{u}_{n-1}^{h\tau})\Big\}\\
&\qquad + \frac{c}{2\tau}\left\{\|\theta_n^{h\tau}\|_Y^2 - \|\theta_{n-1}^{h\tau}\|_Y^2\right\} + \frac{1}{2\tau}\Big\{K(\alpha_n^{h\tau}, \alpha_n^{h\tau}) - K(\alpha_{n-1}^{h\tau}, \alpha_{n-1}^{h\tau})\Big\}\\
&\quad \leq C(\|\boldsymbol{v}_n^{h\tau}\|_H + \|\theta_n^{h\tau}\|_Y + \|\theta_n^{h\tau}\|_Y^2).
\end{aligned}$$

Multiplying the previous estimates by τ and summing up to n we find that

$$\begin{aligned}
&\|\boldsymbol{v}_n^{h\tau}\|_H^2 + a(\boldsymbol{u}_n^{h\tau}, \boldsymbol{u}_n^{h\tau}) + 2G(\alpha_n^{h\tau}, \boldsymbol{u}_n^{h\tau}) + \|\theta_n^{h\tau}\|_Y^2 + K(\alpha_n^{h\tau}, \alpha_n^{h\tau}) \\
&\quad \leq C\tau \sum_{j=1}^{n} (\|\boldsymbol{v}_j^{h\tau}\|_H + \|\theta_j^{h\tau}\|_Y + \|\theta_j^{h\tau}\|_Y^2) \\
&\quad \leq C + C\tau \sum_{j=1}^{n} (\|\boldsymbol{v}_j^{h\tau}\|_H^2 + \|\theta_j^{h\tau}\|_Y^2).
\end{aligned}$$

Finally, taking into account again assumption (18) we find that

$$a(\boldsymbol{u}_n^{h\tau}, \boldsymbol{u}_n^{h\tau}) + 2G(\alpha_n^{h\tau}, \boldsymbol{u}_n^{h\tau}) + K(\alpha_n^{h\tau}, \alpha_n^{h\tau}) \geq C(\|\nabla \boldsymbol{u}_n^{h\tau}\|_Q^2 + \|\nabla \alpha_n^{h\tau}\|_H^2),$$

and using a discrete version of Gronwall's inequality we deduce the desired stability property.

Now, we will prove a main error estimates result.

Theorem 3 *Let the assumptions of Theorem 2 hold. Under additional assumption (18), if we denote by $(\boldsymbol{u}, \boldsymbol{v}, \theta, \alpha)$ and $(\boldsymbol{u}^{h\tau}, \boldsymbol{v}^{h\tau}, \theta^{h\tau}, \alpha^{h\tau})$ the respective solutions to problems VP and VP$^{h\tau}$. Then, we have the following a priori error estimates for all $\boldsymbol{w}^h = \{\boldsymbol{w}_n^h\}_{n=0}^N \subset V^h$ and $\eta^h = \{\eta_n^h\}_{n=0}^N \subset E^h$,*

$$\begin{aligned}
&\max_{0\leq n\leq N} \Big\{ \|\boldsymbol{v}_n - \boldsymbol{v}_n^{h\tau}\|_H^2 + \|\theta_n - \theta_n^{h\tau}\|_Y^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau})\|_Q^2 + \|\nabla(\alpha_n - \alpha_n^{h\tau})\|_H^2 \Big\} \\
&\quad \leq C\tau \sum_{j=1}^{N} \Big(\|\dot{\boldsymbol{v}}_j - \delta \boldsymbol{v}_j\|_H^2 + \|\nabla(\dot{\boldsymbol{u}}_j - \delta \boldsymbol{u}_j)\|_Q^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_H^2 + \|\nabla(\boldsymbol{v}_j - \boldsymbol{w}_j^h)\|_Q^2 \\
&\qquad + \|\dot{\theta}_j - \delta\theta_j\|_Y^2 + \|\nabla(\dot{\alpha}_j - \delta\alpha_j)\|_H^2 + \|\theta_j - \eta_j^h\|_Y^2 + \|\nabla(\theta_j - \eta_j^h)\|_H^2 \Big) \\
&\qquad + \frac{C}{\tau} \sum_{j=1}^{N-1} \Big(\|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H^2 + \|\theta_j - \eta_j^h - (\theta_{j+1} - \eta_{j+1}^h)\|_Y^2 \Big) \\
&\qquad + C \max_{0\leq n\leq N} \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 + C \max_{0\leq n\leq N} \|\theta_n - \eta_n^h\|_H^2 + C\Big(\|\boldsymbol{v}^0 - \boldsymbol{v}^{0h}\|_H^2 + \|\theta^0 - \theta^{0h}\|_Y^2 \\
&\qquad + \|\nabla(\boldsymbol{u}^0 - \boldsymbol{u}^{0h})\|_Q^2 + \|\nabla(\alpha^0 - \alpha^{0h})\|_H^2 \Big),
\end{aligned} \tag{34}$$

where C is a positive constant which is independent of the discretization parameters h and τ.

Proof First, we obtain the error estimates for the velocity field. Then, we subtract variational equation (21) at time $t = t_n$ for a test function $\boldsymbol{w} = \boldsymbol{w}^h \in V^h \subset V$ and discrete variational equation (27) to obtain, for all $\boldsymbol{w}^h \in V^h$,

$$\rho(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{h\tau}, \boldsymbol{w}^h)_H + a(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}, \boldsymbol{w}^h) - A(\theta_n - \theta_n^{h\tau}, \boldsymbol{w}^h) + G(\alpha_n - \alpha_n^{h\tau}, \boldsymbol{w}^h) = 0,$$

and so,

$$\begin{aligned}
&\rho(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{h\tau}, \boldsymbol{v}_n - \boldsymbol{v}_n^{h\tau})_H + a(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}, \boldsymbol{v}_n - \boldsymbol{v}_n^{h\tau}) - A(\theta_n - \theta_n^{h\tau}, \boldsymbol{v}_n - \boldsymbol{v}_n^{h\tau}) + G(\alpha_n - \alpha_n^{h\tau}, \boldsymbol{v}_n - \boldsymbol{v}_n^{h\tau}) \\
&\quad = \rho(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{h\tau}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H + a(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}, \boldsymbol{v}_n - \boldsymbol{w}^h) - A(\theta_n - \theta_n^{h\tau}, \boldsymbol{v}_n - \boldsymbol{w}^h) + G(\alpha_n - \alpha_n^{h\tau}, \boldsymbol{v}_n - \boldsymbol{w}^h).
\end{aligned}$$

If we denote by $\delta \boldsymbol{v}_n = (\boldsymbol{v}_n - \boldsymbol{v}_{n-1})/\tau$ and $\delta \boldsymbol{u}_n = (\boldsymbol{u}_n - \boldsymbol{u}_{n-1})/\tau$, taking into account that

$$\begin{aligned}
&(\delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{h\tau}, \boldsymbol{v}_n - \boldsymbol{v}_n^{h\tau})_H \geq \frac{1}{2\tau} \left\{ \|\boldsymbol{v}_n - \boldsymbol{v}_n^{h\tau}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{h\tau}\|_H^2 \right\}, \\
&a(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}, \delta \boldsymbol{u}_n - \delta \boldsymbol{u}_n^{h\tau}) = \frac{1}{2\tau} \Big\{ a(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}, \boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}) - a(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{h\tau}, \boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{h\tau}) \\
&\qquad + a(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau} - (\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{h\tau}), \boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau} - (\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{h\tau})) \Big\},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{\rho}{2\tau}\left\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{h\tau}\|_H^2\right\}\\
&\qquad+\frac{1}{2\tau}\Big\{a(\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau},\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})-a(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau},\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau})\\
&\qquad+a(\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau}),\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau}))\Big\}\\
&\quad\leq C\Big(\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H^2+\|\nabla(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n)\|_Q^2+\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_H^2+\|\nabla(\boldsymbol{v}_n-\boldsymbol{w}^h)\|_Q^2\\
&\qquad+\|\theta_n-\theta_n^{h\tau}\|_Y^2+\|\nabla(\alpha_n-\alpha_n^{h\tau})\|_H^2\Big)+A(\theta_n-\theta_n^{h\tau},\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau})\\
&\qquad-G(\alpha_n-\alpha_n^{h\tau},\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau})+(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{h\tau},\boldsymbol{v}_n-\boldsymbol{w}^h)_H.
\end{aligned}\tag{35}$$

Now, we will derive the error estimates on the temperature field. Then, we subtract variational equation (22) at time $t=t_n$ for a test function $\eta=\eta^h\in E^h\subset E$ and discrete variational equation (28) to obtain, for all $\eta^h\in E^h$,

$$\begin{aligned}
&c(\dot\theta_n-\delta\theta_n^{h\tau},\eta^h)_Y+b(\theta_n-\theta_n^{h\tau},\eta^h)+A(\eta^h,\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau})+K(\alpha_n-\alpha_n^{h\tau},\eta^h)\\
&\qquad+B(\theta_n-\theta_n^{h\tau},\eta^h)-G(\eta^h,\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})=0,
\end{aligned}$$

and so we have, for all $\eta^h\in E^h$,

$$\begin{aligned}
&c(\dot\theta_n-\delta\theta_n^{h\tau},\theta_n-\theta_n^{h\tau})_Y+b(\theta_n-\theta_n^{h\tau},\theta_n-\theta_n^{h\tau})+A(\theta_n-\theta_n^{h\tau},\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau})-G(\theta_n-\theta_n^{h\tau},\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})\\
&\qquad+B(\theta_n-\theta_n^{h\tau},\theta_n-\theta_n^{h\tau})+K(\alpha_n-\alpha_n^{h\tau},\theta_n-\theta_n^{h\tau})\\
&\quad=c(\dot\theta_n-\delta\theta_n^{h\tau},\theta_n-\eta^h)_Y+b(\theta_n-\theta_n^{h\tau},\theta_n-\eta^h)+A(\theta_n-\eta^h,\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau})\\
&\qquad-G(\theta_n-\eta^h,\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})+B(\theta_n-\theta_n^{h\tau},\theta_n-\eta^h)+K(\alpha_n-\alpha_n^{h\tau},\theta_n-\eta^h).
\end{aligned}$$

Denoting by $\delta\theta_n=(\theta_n-\theta_{n-1})/\tau$ and $\delta\alpha_n=(\alpha_n-\alpha_{n-1})/\tau$, keeping in mind that

$$\begin{aligned}
&(\delta\theta_n-\delta\theta_n^{h\tau},\theta_n-\theta_n^{h\tau})_Y\geq\frac{1}{2\tau}\left\{\|\theta_n-\theta_n^{h\tau}\|_Y^2-\|\theta_{n-1}-\theta_{n-1}^{h\tau}\|_Y^2\right\},\\
&b(\theta_n-\theta_n^{h\tau},\theta_n-\theta_n^{h\tau})\geq C\|\nabla(\theta_n-\theta_n^{h\tau})\|_H^2,\\
&K(\alpha_n-\alpha_n^{h\tau},\delta\alpha_n-\delta\alpha_n^{h\tau})\\
&\qquad=\frac{1}{2\tau}\Big\{K(\alpha_n-\alpha_n^{h\tau},\alpha_n-\alpha_n^{h\tau})-K(\alpha_n-\alpha_{n-1}^{h\tau},\alpha_{n-1}-\alpha_{n-1}^{h\tau})\\
&\qquad\quad+K(\alpha_n-\alpha_n^{h\tau}-(\alpha_{n-1}-\alpha_{n-1}^{h\tau}),\alpha_n-\alpha_n^{h\tau}-(\alpha_{n-1}-\alpha_{n-1}^{h\tau}))\Big\},\\
&B(\theta_n-\theta_n^{h\tau},\theta_n-\theta_n^{h\tau})\leq C\|\theta_n-\theta_n^{h\tau}\|_Y^2+\epsilon\|\nabla(\theta_n-\theta_n^{h\tau})\|_H^2,\\
&|A(\theta_n-\eta^h,\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau})|\leq C(\|\nabla(\theta_n-\eta^h)\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau}\|_H^2),
\end{aligned}$$

where $\epsilon>0$ is assumed again small enough, it follows that

$$\begin{aligned}
&\frac{1}{2\tau}\left\{\|\theta_n-\theta_n^{h\tau}\|_Y^2-\|\theta_{n-1}-\theta_{n-1}^{h\tau}\|_Y^2\right\}\\
&\qquad+\frac{1}{2\tau}\Big\{K(\alpha_n-\alpha_n^{h\tau},\alpha_n-\alpha_n^{h\tau})-K(\alpha_{n-1}-\alpha_{n-1}^{h\tau},\alpha_n-\alpha_{n-1}^{h\tau})\\
&\qquad+K(\alpha_n-\alpha_n^{h\tau}-(\alpha_{n-1}-\alpha_{n-1}^{h\tau}),\alpha_n-\alpha_n^{h\tau}-(\alpha_{n-1}-\alpha_{n-1}^{h\tau}))\Big\}\\
&\quad\leq C\Big(\|\dot\theta_n-\delta\theta_n\|_Y^2+\|\nabla(\dot\alpha_n-\delta\alpha_n)\|_H^2+\|\theta_n-\eta^h\|_Y^2+\|\nabla(\theta_n-\eta^h)\|_H^2\\
&\qquad+\|\theta_n-\theta_n^{h\tau}\|_Y^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau}\|_H^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})\|_Q^2\Big)\\
&\qquad-A(\theta_n-\theta_n^{h\tau},\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau})-G(\theta_n-\theta_n^{h\tau},\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})+(\delta\theta_n-\delta\theta_n^{h\tau},\theta_n-\eta^h)_Y.
\end{aligned}\tag{36}$$

Combining both estimates (35) and (36) we have

$$
\begin{aligned}
&\frac{C}{2\tau}\left\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{h\tau}\|_H^2\right\}+\frac{C}{2\tau}\left\{\|\theta_n-\theta_n^{h\tau}\|_Y^2-\|\theta_{n-1}-\theta_{n-1}^{h\tau}\|_Y^2\right\}\\
&\qquad+\frac{1}{2\tau}\Big\{a(\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau},\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})-a(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau},\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau})\\
&\qquad+a(\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau}),\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau}))\Big\}\\
&\qquad+\frac{1}{2\tau}\Big\{K(\alpha_n-\alpha_n^{h\tau},\alpha_n-\alpha_n^{h\tau})-K(\alpha_{n-1}-\alpha_{n-1}^{h\tau},\alpha_n-\alpha_{n-1}^{h\tau})\\
&\qquad+K(\alpha_n-\alpha_n^{h\tau}-(\alpha_{n-1}-\alpha_{n-1}^{h\tau}),\alpha_n-\alpha_n^{h\tau}-(\alpha_{n-1}-\alpha_{n-1}^{h\tau}))\Big\}\\
&\qquad+G(\delta\alpha_n-\delta\alpha_n^{h\tau},\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})+G(\alpha_n-\alpha_n^{h\tau},\delta\boldsymbol{u}_n-\delta\boldsymbol{u}_n^{h\tau})\\
&\quad\le C\Big(\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H^2+\|\nabla(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n)\|_Q^2+\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_H^2+\|\nabla(\boldsymbol{v}_n-\boldsymbol{w}^h)\|_Q^2\\
&\qquad+\|\theta_n-\theta_n^{h\tau}\|_Y^2+\|\nabla(\alpha_n-\alpha_n^{h\tau})\|_H^2+\|\dot\theta_n-\delta\theta_n\|_Y^2+\|\nabla(\dot\alpha_n-\delta\alpha_n)\|_H^2\\
&\qquad+\|\theta_n-\eta^h\|_Y^2+\|\nabla(\theta_n-\eta^h)\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau}\|_H^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})\|_Q^2\\
&\qquad+(\delta\theta_n-\delta\theta_n^{h\tau},\theta_n-\eta^h)_Y+(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{h\tau},\boldsymbol{v}_n-\boldsymbol{w}^h)_H\Big).
\end{aligned}
$$

Observing that

$$
\begin{aligned}
&G(\alpha_n-\alpha_n^{h\tau},\delta\boldsymbol{u}_n-\delta\boldsymbol{u}_n^{h\tau})+G(\delta\alpha_n-\delta\alpha_n^{h\tau},\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})\\
&\qquad=\frac{1}{\tau}\Big\{G(\alpha_n-\alpha_n^{h\tau},\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})-G(\alpha_{n-1}-\alpha_{n-1}^{h\tau},\boldsymbol{u}_n-\boldsymbol{u}_{n-1}^{h\tau})\\
&\qquad\quad+G(\alpha_n-\alpha_n^{h\tau}-(\alpha_{n-1}-\alpha_{n-1}^{h\tau}),\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau}))\Big\},
\end{aligned}
$$

and that using assumption (18) we have

$$
\begin{aligned}
&a(\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau}),\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau}))\\
&\qquad+K(\alpha_n-\alpha_n^{h\tau}-(\alpha_{n-1}-\alpha_{n-1}^{h\tau}),\alpha_n-\alpha_n^{h\tau}-(\alpha_{n-1}-\alpha_{n-1}^{h\tau}))\\
&\qquad+2G(\alpha_n-\alpha_n^{h\tau}-(\alpha_n-\alpha_{n-1}^{h\tau}),\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau}))\ge 0,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{C}{2\tau}\left\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{h\tau}\|_H^2\right\}+\frac{C}{2\tau}\left\{\|\theta_n-\theta_n^{h\tau}\|_Y^2-\|\theta_{n-1}-\theta_{n-1}^{h\tau}\|_Y^2\right\}\\
&\qquad+\frac{1}{2\tau}\Big\{a(\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau},\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})-a(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau},\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{h\tau})\Big\}\\
&\qquad+\frac{1}{2\tau}\Big\{K(\alpha_n-\alpha_n^{h\tau},\alpha_n-\alpha_n^{h\tau})-K(\alpha_{n-1}-\alpha_{n-1}^{h\tau},\alpha_n-\alpha_{n-1}^{h\tau})\Big\}\\
&\qquad+\frac{1}{\tau}\Big\{G(\alpha_n-\alpha_n^{h\tau},\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})-G(\alpha_{n-1}-\alpha_{n-1}^{h\tau},\boldsymbol{u}_n-\boldsymbol{u}_{n-1}^{h\tau})\Big\}\\
&\quad\le C\Big(\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H^2+\|\nabla(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n)\|_Q^2+\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_H^2+\|\nabla(\boldsymbol{v}_n-\boldsymbol{w}^h)\|_Q^2\\
&\qquad+\|\theta_n-\theta_n^{h\tau}\|_Y^2+\|\nabla(\alpha_n-\alpha_n^{h\tau})\|_H^2+\|\dot\theta_n-\delta\theta_n\|_Y^2+\|\nabla(\dot\alpha_n-\delta\alpha_n)\|_H^2\\
&\qquad+\|\theta_n-\eta^h\|_Y^2+\|\nabla(\theta_n-\eta^h)\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{h\tau}\|_H^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{h\tau})\|_Q^2\\
&\qquad+(\delta\theta_n-\delta\theta_n^{h\tau},\theta_n-\eta^h)_Y+(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{h\tau},\boldsymbol{v}_n-\boldsymbol{w}^h)_H\Big).
\end{aligned}
$$

Multiplying the previous estimates by τ and summing up to n, we have

$$\begin{aligned}
&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{h\tau}\|_H^2 + \|\theta_n - \theta_n^{h\tau}\|_Y^2 + a(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}, \boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}) + K(\alpha_n - \alpha_n^{h\tau}, \alpha_n - \alpha_n^{h\tau}) \\
&\qquad +2G(\alpha_n - \alpha_n^{h\tau}, \boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}) \\
&\quad \leq C\tau \sum_{j=1}^{n} \Big(\|\dot{\boldsymbol{v}}_j - \delta \boldsymbol{v}_j\|_H^2 + \|\nabla(\dot{\boldsymbol{u}}_j - \delta \boldsymbol{u}_j)\|_Q^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_H^2 + \|\nabla(\boldsymbol{v}_j - \boldsymbol{w}_j^h)\|_Q^2 \\
&\qquad + \|\theta_j - \theta_j^{h\tau}\|_Y^2 + \|\nabla(\alpha_j - \alpha_j^{h\tau})\|_H^2 + \|\dot{\theta}_j - \delta\theta_j\|_Y^2 + \|\nabla(\dot{\alpha}_j - \delta\alpha_j)\|_H^2 \\
&\qquad + \|\theta_j - \eta_j^h\|_Y^2 + \|\nabla(\theta_j - \eta_j^h)\|_H^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{h\tau}\|_H^2 + \|\nabla(\boldsymbol{u}_j - \boldsymbol{u}_j^{h\tau})\|_Q^2 \\
&\qquad + (\delta\theta_j - \delta\theta_j^{h\tau}, \theta_j - \eta_j^h)_Y + (\delta \boldsymbol{v}_j - \delta \boldsymbol{v}_j^{h\tau}, \boldsymbol{v}_j - \boldsymbol{w}_j^h)_H \Big) \\
&\qquad + C\Big(\|\boldsymbol{v}^0 - \boldsymbol{v}^{0h}\|_H^2 + \|\theta^0 - \theta^{0h}\|_Y^2 + \|\nabla(\boldsymbol{u}^0 - \boldsymbol{u}^{0h})\|_Q^2 + \|\nabla(\alpha^0 - \alpha^{0h})\|_H^2 \Big).
\end{aligned}$$

Finally, using again assumption (18) we find that

$$\begin{aligned}
&a(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}, \boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}) + K(\alpha_n - \alpha_n^{h\tau}, \alpha_n - \alpha_n^{h\tau}) + 2G(\alpha_n - \alpha_n^{h\tau}, \boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau}) \\
&\quad \geq C(\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau})\|_Q^2 + \|\nabla(\alpha_n - \alpha_n^{h\tau})\|_H^2),
\end{aligned}$$

and taking into account that

$$\begin{aligned}
\tau \sum_{j=1}^{n} (\delta \boldsymbol{v}_j - \delta \boldsymbol{v}_j^{h\tau}, \boldsymbol{v}_j - \boldsymbol{w}_j^h)_H &= \sum_{j=1}^{n} (\boldsymbol{v}_j - \boldsymbol{v}_j^{h\tau} - (\boldsymbol{v}_{j-1} - \boldsymbol{v}_{j-1}^{h\tau}), \boldsymbol{v}_j - \boldsymbol{w}_j^h)_H \\
&= (\boldsymbol{v}_n - \boldsymbol{v}_n^{h\tau}, \boldsymbol{v}_n - \boldsymbol{w}_n^h)_H + (\boldsymbol{v}^{0h} - \boldsymbol{v}^0, \boldsymbol{v}_1 - \boldsymbol{w}_1^h)_H \\
&\quad + \sum_{j=1}^{n-1} (\boldsymbol{v}_j - \boldsymbol{v}_j^{h\tau}, \boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h))_H, \\
\tau \sum_{j=1}^{n} (\delta\theta_j - \delta\theta_j^{h\tau}, \theta_j - \eta_j^h)_Y &= \sum_{j=1}^{n} (\theta_j - \theta_j^{h\tau} - (\theta_{j-1} - \theta_{j-1}^{h\tau}), \theta_j - \eta_j^h)_Y \\
&= (\theta_n - \theta_n^{h\tau}, \theta_n - \eta_n^h)_Y + (\theta^{0h} - \theta^0, \theta_1 - \eta_1^h)_Y \\
&\quad + \sum_{j=1}^{n-1} (\theta_j - \theta_j^{h\tau}, \theta_j - \eta_j^h - (\theta_{j+1} - \eta_{j+1}^h))_Y,
\end{aligned}$$

using the above estimates and a discrete version of Gronwall's inequality (see again [1]) we conclude the proof.

Error estimates (34) can be used to obtain the convergence order of the approximations given by Problem VP$^{h\tau}$. Thus, as an example, we have the following result which states the linear convergence of the algorithm.

Corollary 1 *Let the assumptions of Theorem 3 hold. Therefore, if we assume the following additional regularity:*

$$\begin{aligned}
&\boldsymbol{u} \in H^3(0,T;H) \cap W^{1,\infty}(0,T;[H^2(B)]^d) \cap H^2(0,T;V), \\
&\theta \in H^2(0,T;Y) \cap L^\infty(0,T;H^2(B)) \cap H^1(0,T;E),
\end{aligned}$$

it follows that the approximations obtained by Problem VP$^{h\tau}$ are linearly convergent; that is, there exists a positive constant C, independent of the discretization parameters h and τ, such that

$$\max_{0\leq n\leq N} \Big\{ \|\boldsymbol{v}_n - \boldsymbol{v}_n^{h\tau}\|_H + \|\theta_n - \theta_n^{h\tau}\|_Y + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau})\|_Q + \|\nabla(\alpha_n - \alpha_n^{h\tau})\|_H \Big\} \leq C(h+\tau).$$

The proof of the above result is done using the classical results on the approximation by finite elements and the regularities of the initial conditions (see, again, [3]).

5 Numerical results

In this final section, we describe the numerical scheme implemented in the commercial code Matlab for solving Problem VP$^{h\tau}$, and we show some numerical examples to demonstrate the accuracy of the approximations and the behaviour of the solution. Moreover, we point out that, for simplicity, we will restrict to the case of homogeneous and isotropic materials and therefore, the operators used in Problem VP$^{h\tau}$ have the following form:

$$
\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v}) &= \int_B \mu u_{i,j} v_{i,j} + (\lambda+\mu) u_{i,i} v_{i,i}\, d\boldsymbol{x}, \quad \forall \boldsymbol{u}=(u_i)_{i=1}^d,\ \boldsymbol{v}=(v_i)_{i=1}^d \in V,\\
b(\theta,\eta) &= \int_B b\theta_{,i}\eta_{,i}\, d\boldsymbol{x}, \quad \forall \theta,\ \eta \in E,\\
A(\theta,\boldsymbol{v}) &= \int_B a\theta v_{i,i}\, d\boldsymbol{x}, \quad \forall \theta \in E,\ \boldsymbol{v}=(v_i)_{i=1}^d \in V,\\
K(\alpha,\eta) &= \int_B k\alpha_{,i}\,\eta_{,i} d\boldsymbol{x}, \quad \forall \alpha,\ \eta \in E,
\end{aligned}
$$

and we remark that operators G and B neglect in this case.

5.1 Numerical scheme

As a first step, given the solution $\boldsymbol{u}_{n-1}^{h\tau}$, $\boldsymbol{v}_{n-1}^{h\tau}$, $\theta_{n-1}^{h\tau}$ and $\alpha_{n-1}^{h\tau}$ at time t_{n-1}, the discrete velocity and the discrete temperature are obtained by solving the following discrete linear system, for all $\boldsymbol{w}^h \in V^h$ and $\eta^h \in E^h$,

$$
\begin{aligned}
&\rho(\boldsymbol{v}_n^{h\tau},\boldsymbol{w}^h)_H + \tau^2\, a(\boldsymbol{v}_n^{h\tau},\boldsymbol{w}^h) - \tau A(\theta_n^{h\tau},\boldsymbol{w}^h) = \rho(\boldsymbol{v}_{n-1}^{h\tau},\boldsymbol{w}^h)_H + \rho\tau\,(\boldsymbol{f}_n,\boldsymbol{w}^h)_H - \tau\, a(\boldsymbol{u}_{n-1}^{h\tau},\boldsymbol{w}^h),\\
&c(\theta_n^{h\tau},\eta^h)_Y + \tau b(\theta_n^{h\tau},\eta^h) + \tau^2\, K(\theta_n^{h\tau},\eta^h) + \tau\, A(\eta^h,\boldsymbol{v}_n^{h\tau}) = c\tau\,(S_n,\eta^h)_Y + c(\theta_{n-1}^{h\tau},\eta^h)_Y + \tau K(\alpha_{n-1}^{h\tau},\eta^h),
\end{aligned}
$$

where the discrete displacements and the discrete thermal temperature are then recovered from the relations:

$$
\boldsymbol{u}_n^{h\tau} = \tau\boldsymbol{v}_n^{h\tau} + \boldsymbol{u}_{n-1}^{h\tau}, \quad \alpha_n^{h\tau} = \tau\theta_n^{h\tau} + \alpha_{n-1}^{h\tau}.
$$

This numerical scheme was implemented on a 3.2 Ghz PC using Matlab and a typical one-dimensional run ($h = \tau = 10^{-3}$) took about 0.66 seconds of CPU time.

5.2 First example: numerical convergence in a 1D problem

As a first example, in order to show the numerical convergence of the algorithm we will solve the following one-dimensional academic problem:

Problem P^{ex}. *Find the displacement field $u : [0,1]\times[0,1] \to \mathbb{R}$ and the thermal displacement $\alpha : [0,1]\times[0,1] \to \mathbb{R}$ such that*

$$
\begin{aligned}
&\ddot{u} = 3u_{xx} - \dot{\alpha}_x + f \quad \textit{in} \quad (0,1)\times(0,1),\\
&\ddot{\alpha} = -\dot{u}_x + \alpha_{xx} + \dot{\alpha}_{xx} \quad \textit{in} \quad (0,1)\times(0,1),\\
&u(0,t) = u(1,t) = \alpha(0,t) = \alpha(1,t) = 0 \quad \textit{for a.e.} \quad t \in (0,1),\\
&u(x,0) = \dot{u}(x,0) = \alpha(x,0) = \dot{\alpha}(x,0) = x(1-x) \quad \textit{for a.e.} \quad x \in (0,1),
\end{aligned}
$$

where the body forces f are given by

$$
f(x,t) = tx(x-1) \quad \text{for all} \quad (x,t) \in (0,1)\times(0,1).
$$

We note that Problem P^{ex} corresponds to the thermoelastic problem (30), (6) and (7) with isotropic and homogeneous materials using the following data:

$$
B=(0,1), \quad T=1, \quad \rho=1, \quad a=1, \quad \mu=2, \quad \lambda=1, \quad c=1, \quad k=1, \quad b=1,
$$

and the initial conditions, for all $x \in (0,1)$,

$$
u^0(x) = v^0(x) = x(x-1), \quad \alpha^0(x) = \theta^0(x) = x(x-1).
$$

The approximation errors estimated by

$$
\max_{0\le n\le N} \Big\{ \|\boldsymbol{v}_n - \boldsymbol{v}_n^{h\tau}\|_H + \|\theta_n - \theta_n^{h\tau}\|_Y + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{h\tau})\|_Q + \|\nabla(\alpha_n - \alpha_n^{h\tau})\|_H \Big\}
$$

$h \downarrow \tau \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.328370	0.293531	0.272486	0.265490	0.262008	0.259928	0.259237
$1/2^4$	0.192888	0.155964	0.133760	0.126380	0.122712	0.120530	0.119808
$1/2^5$	0.134851	0.095453	0.072229	0.064617	0.060844	0.058601	0.057861
$1/2^6$	0.110847	0.068228	0.043496	0.035599	0.031738	0.029454	0.028699
$1/2^7$	0.102256	0.056800	0.030017	0.021646	0.017649	0.015322	0.014558
$1/2^8$	0.099676	0.052722	0.024076	0.015007	0.010772	0.008371	0.007597
$1/2^9$	0.098987	0.051504	0.021826	0.012064	0.007473	0.004939	0.004146
$1/2^{10}$	0.098811	0.051179	0.021120	0.010946	0.006003	0.003265	0.002434
$1/2^{11}$	0.098767	0.051096	0.020927	0.010595	0.005442	0.002486	0.001594
$1/2^{12}$	0.098756	0.051076	0.020877	0.010499	0.005265	0.002167	0.001199
$1/2^{13}$	0.098753	0.051070	0.020864	0.010474	0.005216	0.002058	0.001033

Table 1. Example 1: Numerical errors ($\times 10^2$) for some h and τ.

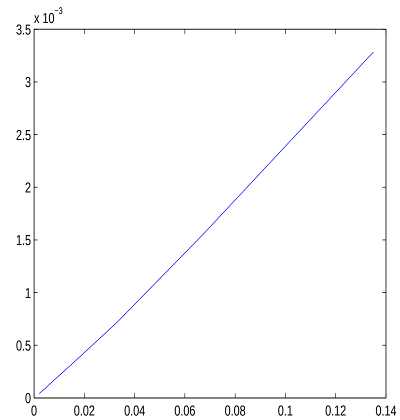

Fig. 1. Example 1: Asymptotic behaviour of the numerical scheme.

are presented in Table 1 (multiplied by 10^2) for several values of the discretization parameters h and τ. Moreover, the evolution of the error depending on the parameter $h+\tau$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 1, seems to be achieved.

If we assume now that there are not volume forces, and we use the final time $T = 80$ and the following data:

$$B = (0,1), \quad \rho = 0.1, \quad a = 0.5, \quad \mu = 1, \quad \lambda = 1, \quad c = 2, \quad k = 1, \quad b = 5,$$

and the initial conditions, for all $x \in (0,1)$,

$$u^0(x) = x(x-1), \quad v^0(x) = \alpha^0(x) = \theta^0(x) = 0,$$

taking the discretization parameters $h = 0.01$ and $\tau = 0.001$, the evolution in time of the discrete energy $E_n^{h\tau}$ defined by

$$E_n^{h\tau} = \frac{1}{2}\Big\{\rho\|v_n^{h\tau}\|_Y^2 + \mu\|(u_n^{h\tau})_x\|_Y^2 + (\lambda+\mu)\|(u_n^{h\tau})_x\|_Y^2 + c\|\theta_n^{h\tau}\|_Y^2 + k\|(\alpha_n^{h\tau})_x\|_Y^2\Big\},$$

is plotted in Fig. 2 in both natural (left) and semi-log (right) scales. As can be seen, it converges to zero and an exponential decay seems to be achieved.

5.3 Second example: temperature generation by a mechanical force

For this second example the square domain $[0,1]\times[0,1]$ is considered, assumed to be clamped on its left part $\{0\}\times[0,1]$ and traction-free on the rest of the boundary. We also suppose that the temperature vanishes on the whole boundary. No heat supply is applied and we use the following expression for the mechanical volume force:

$$\boldsymbol{f}(x,y,t) = (2tx, 2tx) \quad \text{for all} \quad (x,y) \in (0,1)\times(0,1).$$

The following data have been employed in this simulation:

$$B = (0,1)\times(0,1), \quad T = 1, \quad \rho = 0.2, \quad a = 0.5, \quad \mu = 1, \quad \lambda = 1, \quad c = 2, \quad k = 1, \quad b = 5,$$

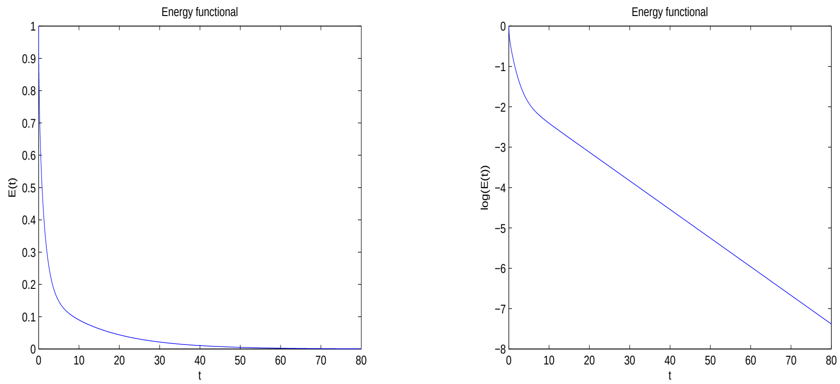

Fig. 2. Example 1: Discrete energy evolution in natural and semi-log scales.

and the initial conditions, for all $(x, y) \in (0,1) \times (0,1)$,

$$\boldsymbol{u}^0(x,y) = \boldsymbol{v}^0(x,y) = (0,0), \quad \alpha^0(x,y) = \theta^0(x,y) = 0.$$

Taking the time discretization parameter $\tau = 0.001$, in Fig. 3 we plot the displacement field in vector arrows on the left-hand side and its norm, at final time, on the right-hand side. As expected, the body moves through the diagonal direction due to the clamping condition and the applied force. The stress field is plotted over the deformed configuration (x10) at final time in Fig. 4. Since we assumed that the body was traction-free on the Neumann boundary, the highest stressed areas concentrate in the interior of the body. Finally, the thermal displacements (left) and the temperature (right) are plotted in Fig. 5 at final time. We note that both neglect on the boundary due to the null boundary conditions and they are produced by the deformation of the body.

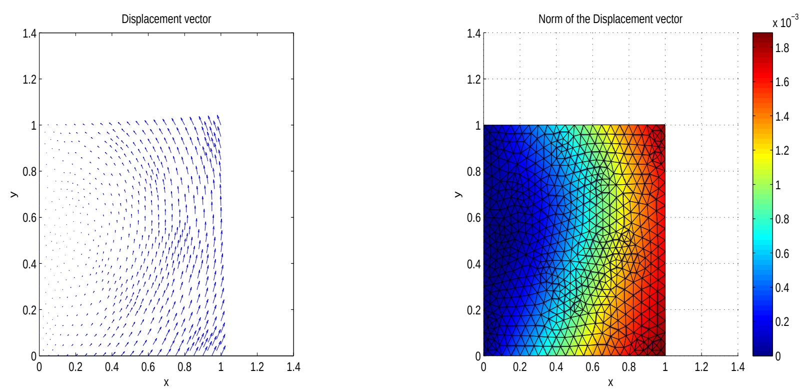

Fig. 3. Example 2: Displacements (left) and its norm (right) at final time.

5.4 Third example: dependence on the thermal coupling coefficient

In this last example, we consider a similar setting than in the previous example. Now, we assume that no volume forces are applied, that the thermal coupling coefficient a varies between 0.01 and 20 and that a traction force $\boldsymbol{f}_F$ acts on the right vertical boundary with the following expression:

$$\boldsymbol{f}_F(x,1,t) = (-t, 0) \quad \text{for all} \quad x \in (0,1).$$

Although this modified problem was not analyzed in the previous sections, it can be done straightforwardly using the same arguments.

Taking the time discretization parameter $\tau = 0.001$ and value $a = 1$ for the thermal coupling coefficient, in Fig. 6 we plot the deformed mesh and the resulting stresses (left) and the temperature (right) at final time. As expected,

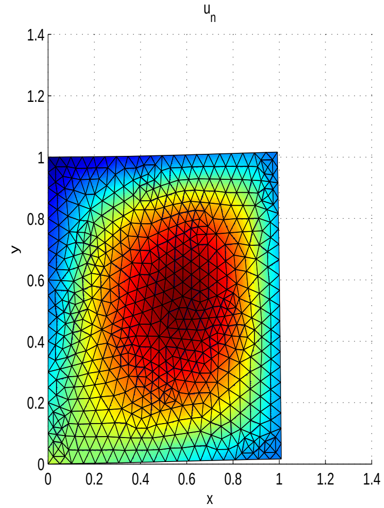

Fig. 4. Example 2: Stresses over the deformed configuration (x10) at final time.

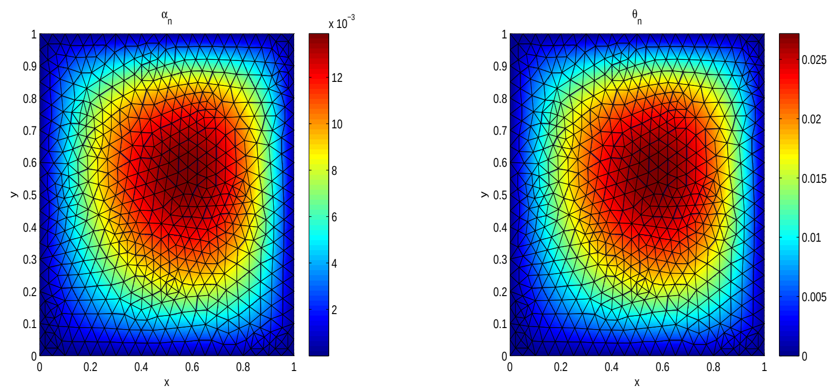

Fig. 5. Example 2: Thermal displacements (left) and temperature (right) at final time.

there is a compression of the body in the longitudinal axis, and an extension on the vertical one, due to the clamping condition. Moreover, temperature is again generated by this deformation and we note again that it concentrates on the internal part of the body due to the null boundary conditions. Finally, in Fig. 7 we plot the evolution in time of the displacements (left) and the temperature (right) at point $\boldsymbol{x} = (0.5, 0.5)$. The displacements have an oscillating behaviour through the time although they are quite similar, but the temperature increases as parameter a also does.

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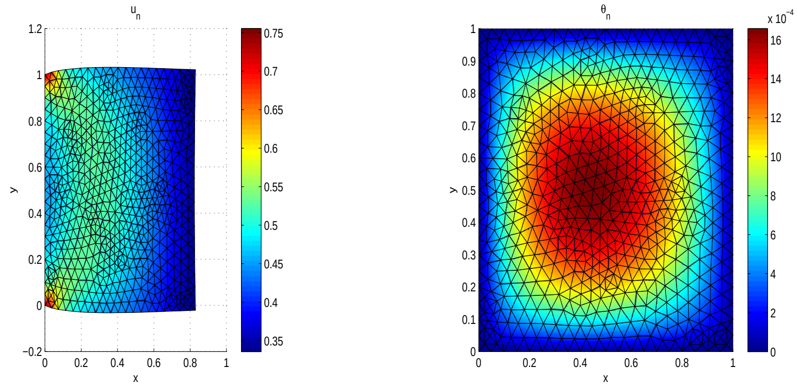

Fig. 6. Example 3: Deformed mesh and stresses (left) and temperature (right) at final time.

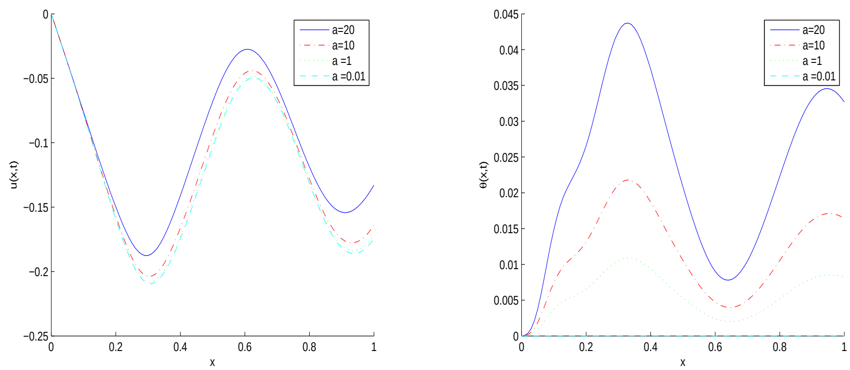

Fig. 7. Example 3: Evolution in time of the displacements (left) and temperature (right) at point $\boldsymbol{x} = (0.5, 0.5)$.

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